Selection principles related to α_i-properties

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Abstract

We investigate selection principles which are motivated by Arhangel'skiĭ's α_i-properties, $i = 1, 2, 3, 4$, and their relations with classical selection principles. It will be shown that they are closely related to the selection principle S_1 and often are equivalent to it.

2000 Mathematics Subject Classification: 54D20, 54B20, 54D55, 54H11.

Keywords: Selection principles, ω-cover, k-cover, γ-cover, γ_k-cover, boundedness of topological groups, upper Fell topology, upper Vietoris topology.

1 Introduction

In this paper we use the usual topological notation and terminology [8] and consider infinite Hausdorff spaces.

Let us fix some more notation and terminology regarding selection principles and families of open covers of a topological space which are necessary for this exposition. For more information in connection with selection principles we refer the interested reader to the survey papers [10], [19], [21].

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of sets of an infinite set X.

The symbol $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection principle:

> For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

*Supported, in part, by MNZŽS RS.

When both $\mathcal{A}$ and $\mathcal{B}$ are the collection $\mathcal{O}$ of open covers of a space X, then $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ defines the classical *Rothberger covering property* (see [16]).

There is an infinite game, denoted $\mathsf{G}_1(\mathcal{A},\mathcal{B})$, corresponding to $\mathsf{S}_1(\mathcal{A},\mathcal{B})$. Two players, ONE and TWO, play a round for each natural number n. In the n–th round ONE chooses a set $A_n \in \mathcal{A}$ and TWO responds by an element b_n from A_n. A play $A_1, b_1; \cdots; A_n, b_n; \cdots$ is won by TWO if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

It is easy to see that if ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$, then the corresponding selection hypothesis $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ is true. However, the converse implication is not always true.

We introduce now new selection principles. The motivation for these definitions is the Arhangel'skiĭ definition of α_i-properties, $i = 1,2,3,4$, introduced in [1]. $\mathcal{A}$ and $\mathcal{B}$ are as above.

Definition 1 The symbol $\alpha_i(\mathcal{A},\mathcal{B})$, $i = 1,2,3,4$, denotes the following selection hypothesis:

> For each sequence $(A_n : n \in \mathbb{N})$ of infinite elements of $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:
>
> $\alpha_1(\mathcal{A},\mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \setminus B$ is finite;
>
> $\alpha_2(\mathcal{A},\mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
>
> $\alpha_3(\mathcal{A},\mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
>
> $\alpha_4(\mathcal{A},\mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is nonempty.

Evidently, if all members of $\mathcal{A}$ are infinite, then

$$\alpha_1(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_2(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_3(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_4(\mathcal{A},\mathcal{B})$$

and

$$\mathsf{S}_1(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_4(\mathcal{A},\mathcal{B}).$$

However, if $\mathcal{A}$ contains finite members, then $\alpha_1(\mathcal{A},\mathcal{B})$ does not not imply $\alpha_2(\mathcal{A},\mathcal{B})$, while $\alpha_3(\mathcal{A},\mathcal{B})$ fails (see [23]).

If for a space X and a point $x \in X$, Σ_x denotes the family of nontrivial sequences in X that converge to x, then X has the Arhangel'skiĭ α_i-property, $i = 1,2,3,4$, if for each $x \in X$ the property $\alpha_i(\Sigma_x,\Sigma_x)$, $i = 1,2,3,4$, holds.

It is known that the four properties $\alpha_i(\Sigma_x,\Sigma_x)$ are different from each other [1], [14] and that the same holds in topological groups [20], [15]. However, it was shown in [18] that in function spaces $\mathsf{C}_p(X)$ and in some hyperspaces [7] the properties α_2, α_3 and α_4 are equivalent to each other and

to the corresponding S_1 property. We shall see here that for some classes $\mathcal{A}$ and $\mathcal{B}$ the properties $\alpha_2(\mathcal{A}, \mathcal{B})$, $\alpha_3(\mathcal{A}, \mathcal{B})$ and $\alpha_4(\mathcal{A}, \mathcal{B})$ are closely related (and often equivalent) to $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$.

Let X be a topological space, $x \in X$, $A \subset X$. Then we use the following notation.

- $\mathcal{O}$: the collection of open covers of X;
- Ω: the collection of ω-covers of X;
- $\mathcal{K}$: the collection of k-covers of X;
- Γ: the collection of γ-covers;
- Γ_k: the collection of γ_k-covers;
- Ω_x: the set $\{A \subset X \setminus \{x\} : x \in \overline{A}\}$;
- Σ_x: the set of all nontrivial sequences in X that converge to x.

An open cover $\mathcal{U}$ of a space X is called an *ω-cover* (a *k-cover*) if every finite (compact) subset of X is contained in a member of $\mathcal{U}$ and X is not a member of $\mathcal{U}$ (i.e. we consider non-trivial covers).

An open cover $\mathcal{U}$ of X is said to be a *γ-cover* (*γ_k-cover*) if it is infinite, and for each finite (compact) subset A of X the set $\{U \in \mathcal{U} : A \nsubseteq U\}$ is finite.

Observe that each infinite subset of a γ-cover (γ_k-cover) is still a γ-cover (γ_k-cover). So, we may suppose that such covers are countable. Each finite (compact) subset of an infinite (non-compact) space belongs to infinitely many elements of an ω-cover (k-cover) of the space.

Recall that a space X is said to be *ω-Lindelöf* (*k-Lindelöf*) if every ω-cover (k-cover) of X contains a countable ω-cover (k-cover).

2 General results

In this section we discuss covering and closure-type properties $\alpha_i(\mathcal{A}, \mathcal{B})$, $i = 2, 3, 4$, in topological spaces and identify some classes $\mathcal{A}$ and $\mathcal{B}$ for which these properties are equivalent to $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$.

We have already mentioned that every space X satisfying the Rothberger covering property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ satisfies also $\alpha_4(\mathcal{O}, \mathcal{O})$. The real line $\mathbb{R}$ satisfies all the properties $\alpha_i(\mathcal{O}, \mathcal{O})$, $i = 2, 3, 4$, but $\mathbb{R}$ does not have the Rothberger property.

However, we have the following result.

Theorem 2 *For an ω-Lindelöf space X the following are equivalent:*

(1) *X satisfies $\alpha_2(\Omega, \Gamma)$;*

(2) *X satisfies $\alpha_3(\Omega, \Gamma)$;*

(3) *X satisfies $\alpha_4(\Omega, \Gamma)$;*

(4) *X satisfies $\mathsf{S}_1(\Omega, \Gamma)$.*

Proof. (3) $\Rightarrow$ (4): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of X. Assume that for each $n \in \mathbb{N}$ we have $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ define

$$\mathcal{V}_n = \{U_{1,m_1} \cap \cdots \cap U_{n,m_n} : n < m_1 < m_2 < \cdots < m_n, U_{i,m_i} \in \mathcal{U}_i, i \leq n\} \setminus \{\emptyset\}.$$

Then each $\mathcal{V}_n$ is an ω-cover of X. By (3) and the fact that each infinite subset of a γ-cover is also a γ-cover, there is an increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ and a γ-cover $\mathcal{V} = \{V_{n_i} : i \in \mathbb{N}\}$ such that for each $i \in \mathbb{N}$, $V_{n_i} \in \mathcal{V}_{n_i}$. Let for each $i \in \mathbb{N}$,

$$V_{n_i} = U_{1,m_1} \cap \cdots \cap U_{n_i, m_{n_i}},\ U_{j,m_j} \in \mathcal{U}_j,\ j \leq n_i.$$

Put $n_0 = 0$. For each $i \geq 0$ and each n with $n_i < n \leq n_{i+1}$ let H_n be the n-th coordinate in the chosen representation of $V_{n_{i+1}}$:

$$H_n = U_{n, m_{n_{i+1}}}.$$

For each $n \in \mathbb{N}$, $H_n \in \mathcal{U}_n$ and the set $\mathcal{H} := \{H_n : n \in \mathbb{N}\}$ is a γ-cover of X because $\mathcal{V}$ is a refinement of $\mathcal{H}$, and $X \notin \mathcal{H}$. Therefore, X satisfies $\mathsf{S}_1(\Omega, \Gamma)$.

(4) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of X and let for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. We shall use the fact that $\mathsf{S}_1(\Omega, \Gamma)$ is equivalent to ONE has no winning strategy in the game $\mathsf{G}_1(\Omega, \Gamma)$ on X [17]. Define the following strategy σ for ONE. ONE's first move is $\sigma(\emptyset) = \mathcal{U}_1$. Assuming that the set $U_{1,m_{i_1}} \in \mathcal{U}_1$ is TWO's response, ONE plays $\sigma(U_{1,m_{i_1}})$ to be $\mathcal{V}(1, m_{i_1}) = \{U_{1,m} : m > m_{i_1}\}$, still an ω-cover of X. If TWO now chooses a set $U_{1,m_{i_2}} \in \mathcal{V}(1, m_{i_1})$, ONE plays $\sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}}) = \mathcal{V}(1, m_{i_2}) = \{U_{1,m} : m > m_{i_2}\}$ which is still an ω-cover of X. Then TWO chooses a set $U_{1,m_{i_3}} \in \sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}})$. And so on.

[Note: For each $n \in \mathbb{N}$ and each $\mathcal{U}_n$ moves of ONE form a new sequence of ω-covers and ensure that from each $\mathcal{U}_n$ TWO chooses infinitely many elements.]

Since σ is not a winning strategy for ONE, consider a σ-play

$$\sigma(\emptyset), U_{1,m_{i_1}}; \sigma(U_{1,m_{i_1}}), U_{1,m_{i_2}}; \sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}}), U_{1,m_{i_3}}; \cdots$$

lost by ONE. That means that the sequence $\mathcal{W}$ consisting of TWO's moves is a γ-cover of X. As it contains infinitely many elements from each $\mathcal{U}_n$, $n \in \mathbb{N}$, $\mathcal{W}$ witnesses for the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ that X has property $\alpha_2(\Omega, \Gamma)$. $\square$

Similarly to the proof of Theorem 2 we can prove the following two theorems. For that we use:

(i) X satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma)$ iff ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{K}, \Gamma)$ on X (see [6]).

(ii) X satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$ iff ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{K}, \Gamma_k)$ on X (see [12]).

Theorem 3 *For a k-Lindelöf non-compact space X, the properties $\alpha_2(\mathcal{K}, \Gamma)$, $\alpha_3(\mathcal{K}, \Gamma)$, $\alpha_4(\mathcal{K}, \Gamma)$ and $\mathsf{S}_1(\mathcal{K}, \Gamma)$ are equivalent.*

Theorem 4 *For a k-Lindelöf non-compact space X, the properties $\alpha_2(\mathcal{K}, \Gamma_k)$, $\alpha_3(\mathcal{K}, \Gamma_k)$, $\alpha_4(\mathcal{K}, \Gamma_k)$ and $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$ are equivalent.*

We also have the following results.

Theorem 5 *For a space X and $\mathcal{B} \in \{\Gamma, \Gamma_k\}$ the following statements are equivalent:*

(1) *X satisfies $\alpha_2(\Gamma_k, \mathcal{B})$;*

(2) *X satisfies $\alpha_3(\Gamma_k, \mathcal{B})$;*

(3) *X satisfies $\alpha_4(\Gamma_k, \mathcal{B})$;*

(4) *X satisfies $\mathsf{S}_1(\Gamma_k, \mathcal{B})$.*

Proof. We have to prove only (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1).

(3) $\Rightarrow$ (4): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of γ_k-covers of X. Enumerate every $\mathcal{U}_n$ bijectively as $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. For all $n, m \in \mathbb{N}$ define

$$V_{n,m} = U_{1,m} \cap U_{2,m} \cap \cdots \cap U_{n,m}.$$

Then for each n the set $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$ is a γ_k-cover of X, because $\mathcal{U}_n$'s are γ_k-covers. By (4) applied to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ there is an

increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ and a cover $\mathcal{V} = (V_{n_i,m_i} : i \in \mathbb{N}) \in \mathcal{B}$ such that for each $i \in \mathbb{N}$, $V_{n_i,m_i} \in \mathcal{V}_{n_i}$. Put $n_0 = 0$. For each $i \geq 0$, each j with $n_i < j \leq n_{i+1}$ and each $V_{n_{i+1},m_{i+1}} = U_{1,m_{i+1}} \cap \cdots \cap U_{n_{i+1},m_{i+1}}$ put

$$H_j = U_{j,m_{i+1}}.$$

For each $j \in \mathbb{N}$, $H_j \in \mathcal{U}_j$ and the set $\{H_j : j \in \mathbb{N}\}$ is in $\mathcal{B}$ because this set is refined by $\mathcal{V}$ which is in $\mathcal{B}$. So, X satisfies $\mathsf{S}_1(\Gamma_k, \mathcal{B})$.

(4) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of γ_k-covers of X. Suppose that for each $n \in \mathbb{N}$, we have $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Choose an increasing sequence $k_1 < k_2 < \cdots < k_p < \cdots$ of positive integers and for each n and each k_i consider $\mathcal{V}(n, k_i) := \{U_{n,m} : m \geq k_i\}$. Then each $\mathcal{V}(n, k_i)$, $n, i \in \mathbb{N}$, is a γ_k-cover of X. Apply now (1) to the sequence $(\mathcal{V}(n, k_i) : i \in \mathbb{N}, n \in \mathbb{N})$ from Γ_k and find a sequence $(V_{n,k_i} : i, n \in \mathbb{N})$ such that for each $(n, i) \in \mathbb{N} \times \mathbb{N}$, $V_{n,k_i} \in \mathcal{V}(n, k_i)$ and the set $\mathcal{W} := \{V_{n,k_i} : n, i \in \mathbb{N}\} \in \mathcal{B}$. It is easy to see that $\mathcal{W}$ can be chosen in such a way that for each $n \in \mathbb{N}$ the set $\mathcal{U}_n \cap \mathcal{W}$ is infinite. Therefore, $\mathcal{W}$ witnesses for the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ that X has property $\alpha_2(\Gamma_k, \mathcal{B})$. $\square$

Notice that in a similar way one can prove that for a space X the properties $\alpha_2(\Gamma, \Gamma)$, $\alpha_3(\Gamma, \Gamma)$, $\alpha_4(\Gamma, \Gamma)$ and $\mathsf{S}_1(\Gamma, \Gamma)$ are equivalent.

As B. Tsaban observed [23], the property $\alpha_1(\Gamma, \Gamma)$ is strictly stronger than $\mathsf{S}_1(\Gamma, \Gamma)$.

3 Applications to topological groups

Let $(G, \cdot, \tau)$ be a topological group with the neutral element e and let $\mathcal{B}_e$ be a local base at e. For each $U \in \mathcal{B}_e$ with $U \neq G$ define

$$\begin{gathered} o(U) = \{x \cdot U : x \in G\}, \\ \mathcal{O}(e) = \{o(U) : U \in \mathcal{B}_e\}; \\ \omega(U) = \{F \cdot U : F \in \mathbb{F}(G)\}, \\ \Omega(e) = \{\omega(U) : U \in \mathcal{B}_e \text{ and there is no } F \in \mathbb{F}(G) \text{ with } F \cdot U = G\}; \\ k(U) = \{K \cdot U : K \in \mathbb{K}(G)\}, \\ \mathcal{K}(e) = \{k(U) : U \in \mathcal{B}_e \text{ and there is no } K \in \mathbb{K}(G) \text{ with } K \cdot U = G\}. \end{gathered}$$

Then clearly, $\mathcal{O}(e) \subset \mathcal{O}$, $\Omega(e) \subset \Omega$; $\mathcal{K}(e) \subset \mathcal{K}$.

In [2] (see also [9], [13], [22]) Menger-bounded, Rothberger-bounded and Hurewicz-bounded topological groups have been studied. A topological group G is *Menger-bounded* (*Rothberger-bounded*, *Hurewicz-bounded*) if it satisfies the selection principle $\mathsf{S}_1(\Omega(e), \mathcal{O})$ ($\mathsf{S}_1(\mathcal{O}(e), \mathcal{O})$, $\mathsf{S}_1(\Omega(e), \Gamma)$).

We have the following results. Their proofs are similar, so we prove only the first of them.

Theorem 6 *For a topological group G the following are equivalent:*

(1) *G satisfies $\alpha_4(\Omega(e), \Gamma)$;*

(2) *G satisfies $\mathsf{S}_1(\Omega(e), \Gamma)$;*

(3) *G satisfies $\mathsf{S}_1(\mathcal{K}(e), \Gamma)$.*

Proof. The implications (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are obvious.

(1) $\Rightarrow$ (2): Let $(U_n : n \in \mathbb{N})$ be a sequence of elements of $\mathcal{B}_e$. For each $n \in \mathbb{N}$ let $V_n = U_1 \cap U_2 \cap \cdots \cap U_n$ be a member of $\mathcal{B}_e$. If we now apply (1) to the sequence $(V_n : n \in \mathbb{N})$ we find an increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ and finite sets $F_{n_i} \subset G$, $i \in \mathbb{N}$, so that $\{F_{n_i} \cdot V_{n_i} : n \in \mathbb{N}\}$ is a γ-cover of G. If $n_0 = 0$, then for each positive integer n with $n_{i-1} < n \leq n_i$, $i \in \mathbb{N}$, put $F_n = F_{n_i}$ and U_n to be the n-th component in the representation $U_1 \cap \cdots \cap U_{n_i}$ of V_{n_i}. Evidently, $\{F_n \cdot U_n : n \in \mathbb{N}\}$ is a γ-cover of G, i.e. the sequence $(F_n : n \in \mathbb{N})$ guaranties for $(U_n : n \in \mathbb{N})$ that G satisfies $\mathsf{S}_1(\Omega(e), \Gamma)$.

(3) $\Rightarrow$ (2): Let $(U_n : n \in \mathbb{N})$ be a sequence of elements of $\mathcal{B}_e$. For each n pick a $V_n \in \mathcal{B}_e$ so that $V_n^2 \subset U_n$. By (3) choose a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of G such that $\{K_n \cdot V_n : n \in \mathbb{N}\}$ is a γ-cover of G. Next, for each n pick a finite set F_n in G such that $K_n \subset F_n \cdot V_n$. Then for each n we have $K_n \cdot V_n \subset (F_n \cdot V_n) \cdot V_n \subset F_n \cdot U_n$ and one concludes that $\{F_n \cdot U_n : n \in \mathbb{N}\}$ is a γ-cover of G. $\square$

Notice that $\alpha_2(\Omega(e), \Gamma)$ and $\alpha_3(\Omega(e), \Gamma)$ are also equivalent to the properties listed in the theorem above.

Theorem 7 *For a topological group G the following are equivalent:*

(1) *G satisfies $\alpha_4(\Omega(e), \Gamma_k)$;*

(2) *G satisfies $\mathsf{S}_1(\Omega(e), \Gamma_k)$;*

(3) *G satisfies $\mathsf{S}_1(\mathcal{K}(e), \Gamma_k)$.*

4 $\alpha_i(\mathcal{A},\mathcal{B})$ properties in hyperspaces

In this section we consider $\alpha_i(\mathcal{A},\mathcal{B})$ properties, $i = 2, 3, 4$, in hyperspaces. We begin with some definitions that we need.

For a (Hausdorff) space X by 2^X we denote the family of all closed subsets of X. $\mathbb{K}(X)$ is the collection of all non-empty compact subsets of X, and $\mathbb{F}(X)$ denotes the family of all non-empty finite subsets of X. If A is a subset of X and $\mathcal{A}$ a family of subsets of X, then we write

$$A^+ = \{F \in 2^X : F \subset A\},\ \mathcal{A}^+ = \{A^+ : A \in \mathcal{A}\}.$$

Notice that we use the same symbol F to denote a closed subset of X and the point F in 2^X; from the context it will be clear what F is.

The *upper Fell topology* F^+ on 2^X is the topology whose base is the collection

$$\{(K^c)^+ : K \in \mathbb{K}(X)\} \cup \{2^X\},$$

while the *upper Vietoris topology* V^+ has basic sets of the form U^+, U open in X. It is clear that $(\mathbb{K}(X), \mathsf{V}^+)$ and $(\mathbb{F}(X), \mathsf{V}^+)$ are considered as subspaces of $(2^X, \mathsf{V}^+)$.

In [7] it was shown that in $(2^X, \mathsf{F}^+)$ each of Arhangel'skiĭ's α_2, α_3 and α_4 properties is equivalent to $\mathsf{S}_1(\Sigma_E, \Sigma_E)$, $E \in 2^X$. We discuss here some other properties. For similar consideration see [5], [11].

Theorem 8 *If X is a space whose all open subspaces are k-Lindelöf and $E \in 2^X$, then the following statements are equivalent:*

(1) $(2^X, \mathsf{F}^+)$ *satisfies* $\alpha_4(\Omega_E, \Sigma_E)$*;*

(2) $(2^X, \mathsf{F}^+)$ *satisfies* $\mathsf{S}_1(\Omega_E, \Sigma_E)$*.*

Proof. We have to prove only (1) implies (2). Let $(\mathcal{A}_n : n \in \mathbb{N})$ be a sequence of elements of Ω_E. Since each open subspace of X is k-Lindelöf, $(2^X, \mathsf{F}^+)$ has countable tightness (see [4], [3]) and we may assume that for each $n \in \mathbb{N}$, $\mathcal{A}_n$ is countable, say $\mathcal{A}_n = \{A_{n,m} : m \in \mathbb{N}\}$. For each n let $\mathcal{B}_n$ be the collection of all sets of the form

$$A_{1,m_1} \cup A_{2,m_2} \cup \cdots A_{n,m_n},\ A_{i,m_i} \in \mathcal{A}_i,\ i \le n.$$

Then each $\mathcal{B}_n$ belongs to Ω_E. Apply (1) to the sequence $(\mathcal{B}_n : n \in \mathbb{N})$ of elements of Ω_E. There exist an increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ and a sequence $\mathcal{B} := (B_{n_i} : i \in \mathbb{N}) \in \Sigma_E$ such that for each $i \in \mathbb{N}$, $B_{n_i} \in \mathcal{B}_{n_i}$. Put $n_0 = 0$ and define the sequence $(S_n : n \in \mathbb{N})$ in the following manner:

If $i \geq 0$, then for each n with $n_i < n \leq n_{i+1}$ define S_n to be A_{n,m_n} in the chosen representation of $B_{n_{i+1}}$.

Note that for each $n \in \mathbb{N}$, $S_n \in \mathcal{A}_n$ and evidently the sequence $\mathcal{S} := (S_n : n \in \mathbb{N})$ is an element of Σ_E. So, $\mathcal{S}$ is a selector for the original sequence $(\mathcal{A}_n : n \in \mathbb{N})$ showing that $(2^X, \mathsf{F}^+)$ satisfies (2). $\square$

In what follows we shall need the following two simple lemmas. Because their proofs are similar we prove only the first of them.

Lemma 9 *For a space X and an open cover $\mathcal{W}$ of $(\mathbb{K}(X), \mathsf{V}^+)$ the following holds: $\mathcal{W}$ is an ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$ if and only if $\mathcal{U}(\mathcal{W}) := \{U \subset X : U$ is open in X and $U^+ \subset W$ for some $W \in \mathcal{W}_n\}$ is a k-cover of X.*

Proof. Let $\mathcal{W}$ be an ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$ and let K be a compact subset of X. Then there exists $W \in \mathcal{W}$ such that $K \in W$ and consequently there is an open set $U \subset X$ with $K \in U^+ \subset W$. It is understood, $U \in \mathcal{U}(\mathcal{U})$. On the other hand, $K \subset U$, i.e. $\mathcal{U}(\mathcal{W})$ is a k-cover of X.

Conversely, let $\mathcal{U}(\mathcal{W})$ be a k-cover of X and let $\{K_1, \cdots, K_m\}$ be a finite subset of $(\mathbb{K}(X), \mathsf{V}^+)$. Then $K = \bigcup_{i=1}^m K_i$ is a compact subset of X and thus K is contained in some $U \in \mathcal{U}(\mathcal{W})$; pick $W \in \mathcal{W}$ such that $U^+ \subset W$. From $K_i \subset U$ for each $i \leq m$, it follows $\{K_1, \cdots, K_m\} \subset U^+ \subset W$ which just means that $\mathcal{W}$ is an ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$. $\square$

Lemma 10 *For a space X and an open cover $\mathcal{W}$ of $(\mathbb{F}(X), \mathsf{V}^+)$ the following holds: $\mathcal{W}$ is an ω-cover of $(\mathbb{F}(X), \mathsf{V}^+)$ if and only if $\mathcal{U}(\mathcal{W}) := \{U \subset X : U$ is open in X and $U^+ \subset W$ for some $W \in \mathcal{W}_n\}$ is an ω-cover of X.*

We use now the last two lemmas to prove the next two propositions.

Proposition 11 *A space X is k-Lindeloöf if and only if $(\mathbb{K}(X), \mathsf{V}^+)$ is ω-Lindelöf.*

Proof. Let X be a k-Lindelöf space and let $\mathcal{W}$ be an ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$. By Lemma 9 (and notation from that lemma), $\mathcal{U}(\mathcal{W})$ is a k-cover of X. Choose a countable family $\{U_i : i \in \mathbb{N}\} \subset \mathcal{U}(\mathcal{W})$ which is a k-cover of X. For each $i \in \mathbb{N}$ pick $W_i \in \mathcal{W}$ such that $U_i^+ \subset W_i$. Again by Lemma 9 $\{W_i : i \in \mathbb{N}\} \subset \mathcal{W}$ is an ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$.

Let us show the converse. Let $\mathcal{U}$ be a k-cover of X. It is easy to check that $\mathcal{U}^+$ is an ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$. Choose a countable collection $\{U_i^+ : i \in \mathbb{N}\} \subset \mathcal{U}^+$ which is an ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$. Then $\{U_i : i \in \mathbb{N}\} \subset \mathcal{U}$ is a k-cover of X, i.e. X is a k-Lindelöf space. $\square$

Similarly, by using Lemma 10, one obtains

Proposition 12 *A space X is ω-Lindeloöf if and only if $(\mathbb{F}(X), \mathsf{V}^+)$ is ω-Lindelöf.*

Theorem 13 *For a k-Lindelöf space X the following are equivalent:*

(1) $(\mathbb{K}(X), \mathsf{V}^+)$ *satisfies* $\alpha_2(\Omega, \Gamma)$*;*

(2) $(\mathbb{K}(X), \mathsf{V}^+)$ *satisfies* $\alpha_3(\Omega, \Gamma)$*;*

(3) $(\mathbb{K}(X), \mathsf{V}^+)$ *satisfies* $\alpha_4(\Omega, \Gamma)$*;*

(4) $(\mathbb{K}(X), \mathsf{V}^+)$ *satisfies* $\mathsf{S}_1(\Omega, \Gamma)$*;*

(5) X *satisfies* $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$*.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold for any space.

(4) $\Rightarrow$ (1): By Proposition 11 the space $(\mathbb{K}(X), \mathsf{V}^+)$ is ω-Lindelöf. It remains to apply Theorem 2.

(4) $\Rightarrow$ (5): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of k-covers of X. Then $(\mathcal{U}_n^+ : n \in \mathbb{N})$ is a sequence of ω-covers of $(\mathbb{K}(X), \mathsf{V}^+)$. Indeed, fix n and let $\{K_1, \cdots, K_m\}$ be a finite subset of $\mathbb{K}(X)$. Then $K = K_1 \cup \cdots \cup K_m$ is a compact subset of X and thus there is $U \in \mathcal{U}$ with $K \subset U$. This means that for each $i \leq m$, $K_i \subset U$, i.e. $K_i \in U^+$. Therefore $\{K_1, \cdots, K_m\} \subset U^+$ and $\mathcal{U}_n$ is an ω-cover of $\mathcal{K}(X)$. By (4) for each n, choose an element U_n^+ in $\mathcal{U}_n^+$ such that the set $\mathcal{U}^+ = \{U_n^+ : n \in \mathbb{N}\}$ is a γ-cover of $(\mathbb{K}(X), \mathsf{V}^+)$. We prove that $\{U_n : n \in \mathbb{N}\}$ is a γ_k-cover of X. Let K be a compact subset of X. Then there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ we have $K \in U_n^+$, hence $K \subset U_n$. It shows that $\{U_n : n \in \mathbb{N}\}$ is really a γ_k-cover of X, i.e. that (5) holds.

(5) $\Rightarrow$ (4): Let $(\mathcal{W}_n : n \in \mathbb{N})$ be a sequence of ω-covers of $(\mathbb{K}(X), \mathsf{V}^+)$. For each n let

$$\mathcal{U}_n = \{U \subset X : U \text{ is open in } X \text{ and } U^+ \subset W \text{ for some } W \in \mathcal{W}_n\}.$$

By Lemma 9 each $\mathcal{U}_n$ is a k-cover of X. By (5) applied to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ one can find a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and the set $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a γ_k-cover of X. For each $U_n \in \mathcal{U}$ pick an element $W_n \in \mathcal{W}_n$ so that $U_n^+ \subset W_n$. We claim that $\{W_n : n \in \mathbb{N}\}$ is a γ-cover of $(\mathbb{K}(X), \mathsf{V}^+)$ and so it witnesses for $(\mathcal{W}_n : n \in \mathbb{N})$ that (4)

is satisfied. Let $K \in \mathbb{K}(X)$. Then there is n_0 such that for each $n \geq n_0$, $K \subset U_n$, i.e. $K \in U_n^+ \subset W_n$. $\square$

It is not difficult to verify that in a similar way, using Proposition 12 and Theorem 2, one obtains the following theorem.

Theorem 14 *For an ω-Lindelöf space X the following are equivalent:*

(1) $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\alpha_2(\Omega, \Gamma)$*;*

(2) $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\alpha_3(\Omega, \Gamma)$*;*

(3) $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\alpha_4(\Omega, \Gamma)$*;*

(4) $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\mathsf{S}_1(\Omega, \Gamma)$*;*

(5) X *satisfies* $\mathsf{S}_1(\Omega, \Gamma)$.

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